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Electrostatic charges in $v \times B$ fields: the Faraday disk and the rotating sphere

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Abstract. When a conductor moves in a magnetic field, $\boldsymbol{v} \times \boldsymbol{B}$ acts like a distributed source. If the divergence of this vector is not zero, then the divergence of $\boldsymbol{E}$ is not zero either and the conductor carries electrostatic charges whose fields are as important as $\boldsymbol{v} \times \boldsymbol{B}$. The Faraday disk and a conducting sphere rotating in a uniform magnetic field serve as examples. This little known effect plays a fundamental role in magnetohydrodynamic phenomena.

Résumé. A l'intérieur d'un conducteur en mouvement dans un champ magnétique, $\boldsymbol{v} \times \boldsymbol{B}$ agit comme une source distribuée. Si la divergence de ce champ n'est pas nulle, la divergence de $\boldsymbol{E}$ est non-nulle également et le conducteur porte des charges électrostatiques dont le champ est aussi important que $\boldsymbol{v} \times \boldsymbol{B}$. Le disque de Faraday, et une sphère conductrice tournant dans un champ magnétique uniforme, servent à illustrer notre propos. Cet effet peu connu joue un rôle essentiel dans les phénomènes magnétohydrodynamiques.

1. Introduction

Imagine a body that moves at a velocity $\boldsymbol{v}$ in a region where there exist an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$. Then an electric charge Q inside the body feels a force $Q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$. Thus, inside the moving body, the $\boldsymbol{v} \times \boldsymbol{B}$ field acts like the electric field of a distributed source.

We are concerned here with conducting media that move in magnetic fields. We shall see that they carry electrostatic charges whose field is just as important as $\boldsymbol{v} \times \boldsymbol{B}$. Indeed, there are many cases where the two fields cancel each other exactly at every point. The Faraday disk and a conducting sphere rotating in a magnetic field will serve as examples, but this little known effect plays a fundamental role in magnetohydrodynamics. Another article will discuss the function of these electrostatic charges in the dynamo mechanism that generates the Earth's magnetic field.

We first consider a general velocity $\boldsymbol{v}$, but most of our discussion will concern the rotation of rigid bodies, either connected to a stationary circuit or not. We perform our calculations both with respect to a fixed reference frame and with respect to a rotating frame.

2. Electrostatic charges in $v \times B$ fields

It is well known that conductors do not support an electric space charge; any extra charge deposited inside moves out to the periphery almost instantaneously (Lorrain *et al* 1988, p 75). However, few physicists realise that conductors do carry an electric space charge when subjected to a $\boldsymbol{v} \times \boldsymbol{B}$ field whose divergence is not equal to zero. If the conductor is isolated, then it also carries a compensating surface charge.

2.1. Fixed reference frame

Choose a fixed reference frame S. With respect to S, the electric field strength is $\boldsymbol{E}$ and the magnetic flux density $\boldsymbol{B}$. If now a point inside a body of conductivity σ moves at a velocity $\boldsymbol{v}$ with respect to S, then the electric current density at that point, again with respect to S, is

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \qquad (1)$$

The body need not be rigid and all the terms can be both space and time dependent. The $\boldsymbol{v} \times \boldsymbol{B}$ field acts like the electric field of a distributed source (Lorrain *et al* 1988, p 494).

Now take the divergence of both sides. Under steady-state conditions, the divergence of $\boldsymbol{J}$ is zero and, if the medium is homogeneous,

$$\nabla \cdot \boldsymbol{E} = -\nabla \cdot (\boldsymbol{v} \times \boldsymbol{B}). \tag{2}$$

Thus, if the divergence of $\boldsymbol{v} \times \boldsymbol{B}$ is not zero, there exists an electrostatic space charge of density

$$\varrho = \varepsilon_r \varepsilon_0 \nabla \cdot \boldsymbol{E} = -\varepsilon_r \varepsilon_0 \nabla \cdot (\boldsymbol{v} \times \boldsymbol{B}) \tag{3}$$

$$= \varepsilon_r \varepsilon_0 [\boldsymbol{v} \cdot (\nabla \times \boldsymbol{B}) - \boldsymbol{B} \cdot (\nabla \times \boldsymbol{v})]. \tag{4}$$

Observe that the charge density is independent of the conductivity. The relative permittivity of good conductors is not measurable, but it is presumably of the order of three, like that of ordinary dielectrics.

We use ρ for the distance to the axis of rotation, ϱ (varrho) for the space charge density, σ for the conductivity, and ς (varsigma) for the surface charge density.

With an ideal dielectric, $\boldsymbol{J}$ and σ would be both zero, equation (1) would be meaningless, and these expressions for ϱ would not apply.

One author argues that, in conducting media such as plasmas in space, the charge density is in fact zero because of the abundance of charge carriers. He concludes that, under steady-state conditions, $\boldsymbol{E} = 0$. That is of course a misconception; the above equations apply in any medium and $\boldsymbol{E} \neq 0$ if the divergence of $\boldsymbol{v} \times \boldsymbol{B} \neq 0$.

We shall see that, if the conductor is isolated, surface charges compensate for the space charge.

If the fields are time dependent, and if the medium is not uniform, then there still exists an electric space charge, but there are further terms on the right-hand side of equation (4).

It follows from equation (1) that

$$\boldsymbol{E} = -(\boldsymbol{v} \times \boldsymbol{B}) + \boldsymbol{J}/\sigma. \tag{5}$$

The vectors $\boldsymbol{J}$ and $\boldsymbol{B}$ are not independent; they satisfy the Maxwell equation

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \tag{6}$$

assuming that the medium is non-magnetic, and under steady-state conditions.

Now consider a rotating body. If a rigid axisymmetric conducting body rotates at an angular velocity ω about the z axis in any magnetic field, then

$$\boldsymbol{v} = \omega\rho\hat{\boldsymbol{\phi}} \qquad \nabla \times \boldsymbol{v} = 2\omega\hat{\boldsymbol{z}} \tag{7}$$

$$\varrho = \varepsilon_r \varepsilon_0 \omega(\mu_0 \rho J_\phi - 2B_z) \tag{8}$$

where $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{z}}$ are the unit vectors pointing, respectively, in the positive ϕ and z directions. The azimuthal current comes from the rotation of the space charge, and possibly also from an internal source. If there is no internally generated azimuthal current, then the conductor need not be axisymmetric.

Let us calculate the curl of $\boldsymbol{J}$ for a passive rotating conductor that is not necessarily axisymmetric. From equation (1), in a homogeneous medium,

$$\nabla \times \boldsymbol{J} = \sigma[\nabla \times \boldsymbol{E} + \nabla \times (\boldsymbol{v} \times \boldsymbol{B})] \tag{9}$$

$$= \sigma[-\partial \boldsymbol{B}/\partial t + \nabla \times (\boldsymbol{v} \times \boldsymbol{B})]. \tag{10}$$

Under steady-state conditions the time derivative vanishes and

$$\nabla \times \boldsymbol{J} = \sigma\nabla \times (\boldsymbol{v} \times \boldsymbol{B}). \tag{11}$$

Suppose now that our rotating conductor is *not* connected to a stationary circuit through sliding contacts. Suppose also that the magnetic field is both uniform and parallel to the axis of rotation. Both ω and $\boldsymbol{B}$ are constant. Then

$$\nabla \times \boldsymbol{J} = \sigma\nabla \times (\omega\rho\hat{\boldsymbol{\phi}} \times B\hat{\boldsymbol{z}}) \equiv 0. \tag{12}$$

If the magnetic field is not uniform, but axisymmetric, then it turns out that the curl is again zero:

$$\nabla \times \boldsymbol{J} = \boldsymbol{v}(\nabla \cdot \boldsymbol{B}) \equiv 0. \tag{13}$$

Now, in our isolated rotating conductor, currents can only flow around closed circuits. Then the line integral of $\boldsymbol{J}$ around any closed circuit is equal to zero and there are no induced currents:

$$\boldsymbol{J} = 0 \qquad \boldsymbol{E} = -(\boldsymbol{v} \times \boldsymbol{B}). \tag{14}$$

In this instance, the field of the electrostatic charges cancels the $\boldsymbol{v} \times \boldsymbol{B}$ field exactly at every point.

This is not a new result (van Bladel 1984), but it is important for the following reason. Since there are no induced currents in the rotating conductor, the magnetic field is not disturbed and the lines of $\boldsymbol{B}$ are not dragged along by the moving conductor, as one would expect from Alfven's 'theorem' (Alfven and Falthammar 1963). Another paper will discuss this 'theorem' at some length.

If the applied magnetic field is symmetrical about an axis that forms an angle with the axis of rotation, then we can consider it to be composed of an axisymmetric field B_{axi} plus a transverse field $B_x\hat{\boldsymbol{x}}$. Then

$$\nabla \times \boldsymbol{J} = \sigma\nabla \times [\boldsymbol{v} \times (\boldsymbol{B}_{\text{axi}} + B_x\hat{\boldsymbol{x}})] \tag{15}$$

$$= \sigma\nabla \times (\omega\rho B_x\hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{x}}) \tag{16}$$

$$= \sigma\nabla \times (\omega\rho B_x \cos\phi\hat{\boldsymbol{z}}) \tag{17}$$

and there are induced currents in the conductor.

2.2. Rotating reference frame

Rotating reference frames are troublesome in relativity because they are non-Euclidian and because clock rate depends on the distance to the axis of rotation (Møller 1974). Fortunately, the situation here is simple because we need only compare the values of the various variables at, and in the immediate neighbourhood of, a given point in the two frames.

We use unprimed variables in the inertial reference frame S, as above. Primed variables at a given point P' in the rotating reference frame S' apply in the local inertial reference frame S'' occupied momentarily by P' (Møller 1974). Then the relations between the primed and unprimed variables are those of special relativity (Lorrain *et al* 1988, pp 287, 303, 321). We disregard terms of the order of v^2/c^2, where c is the speed of light. Then

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \boldsymbol{J}'/\sigma' \qquad \sigma' = \sigma \tag{18}$$

$$\boldsymbol{J}' = \boldsymbol{J} \qquad \boldsymbol{B}' = \boldsymbol{B}. \tag{19}$$

Also, in a uniform medium,

$$\nabla \times \boldsymbol{J}' = \sigma \nabla \times \boldsymbol{E}' = -\sigma \partial \boldsymbol{B}'/\partial t'. \tag{20}$$

Recalling now that, in an *isolated* moving conductor, there are no induced currents if the curl of the current density is zero, this equation shows that there are induced currents in an *isolated* moving conductor only if the magnetic induction in the reference frame of the conductor is time dependent. This is in agreement with equations (13) and (14).

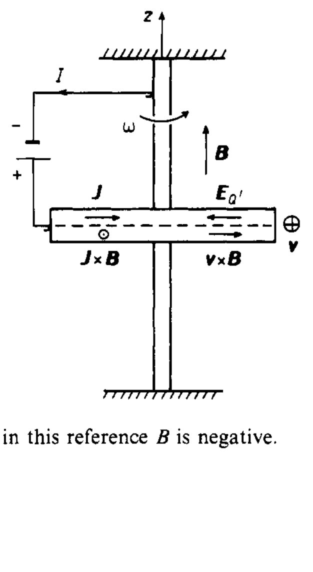
The space charge density is the same in the two reference frames if the azimuthal current results only from the motion of the space charge.

3. The Faraday disk

Figures 1 and 2 show schematic diagrams of a homopolar motor and of a homopolar generator, as viewed from a fixed reference frame (Lorrain *et al* 1988, p 399†). Both devices are also called Faraday disks. We have shown single brushes on the axis and on the rim, but there are in fact continuous brushes all around. Also, we have not shown the coils that generate the applied magnetic fields $\boldsymbol{B}$. At any point in the disks, $v^2 << c^2$.

Assume that the axial magnetic field $\boldsymbol{B}$ is that of a long solenoid, and is thus uniform. This assumption is not realistic, but it makes the calculation simple and instructive. We may set $J_\phi = 0$ in equation (8) because, as we shall see below, the convection current of the rotating space charge generates an opposing axial magnetic field that is negligible. We disregard the magnetic field of the radial current through the disks because it is azimuthal, which makes its $\boldsymbol{v} \times \boldsymbol{B}$ equal to zero. Set $B_z = B$ in equation (8).

Figure 1. Homopolar motor.



† Note that in this reference B is negative.

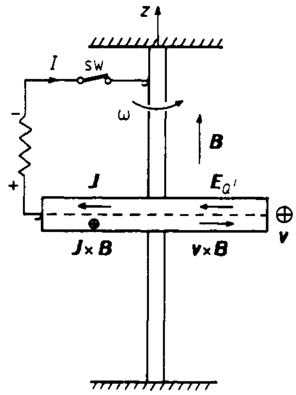


Figure 2. Homopolar generator.

Say B and ω are the same in figures 1 and 2. Then $\boldsymbol{v} \times \boldsymbol{B}$ and ϱ are the same and, from equation (8), the space charge density is uniform and negative:

$$\varrho = -2\varepsilon_r \varepsilon_0 \omega B. \tag{21}$$

This space charge density is independent of the conductivity, except if σ were zero, and it is independent of whether the switch sw in figure 2 is open or closed.

The rotation of this space charge generates an azimuthal convection current of density $\varrho \boldsymbol{v}$ whose magnetic field opposes the above $\boldsymbol{B}$. Since that $\boldsymbol{B}$ is uniform, it is easy to calculate the magnetic induction $\boldsymbol{B}_\varrho$ of this azimuthal convection current at the centre of the disk. Setting R and s equal to the outer radius and to the thickness of the disk, respectively, we find that, for $s << R$,

$$|B_\varrho| = \left| -\varepsilon_r \frac{\omega^2 s R}{c^2} B \right| << B \tag{22}$$

since $\omega^2 sR$ is of the order of v^2, and $v^2/c^2 << 1$ by hypothesis.

Now disregard end effects. This is equivalent to stretching the disk into a long cylinder. This assumption does not invalidate the above approximation. Then

$$E_\varrho = \frac{\pi \rho^2 \varrho}{2\pi \varepsilon_r \varepsilon_0 \rho} = \frac{\varrho}{2\varepsilon_r \varepsilon_0} \rho = -\omega \rho B. \tag{23}$$

Observe that, here,

$$\boldsymbol{E}_\varrho = -(\boldsymbol{v} \times \boldsymbol{B}). \tag{24}$$

The $\boldsymbol{v} \times \boldsymbol{B}$ field 'pumps' conduction electrons in the $-\hat{\rho}$ direction, and their space charge cancels the $\boldsymbol{v} \times \boldsymbol{B}$ field exactly at every point in both figures 1 and 2, whether the switch sw of figure 2 is open or closed.

The simplest case is that of the generator, with sw open. Then

$$J = 0, \qquad E + v \times B = 0 \qquad E = E_Q. \tag{25}$$

A surface charge of density $\varepsilon_r \varepsilon_0 \omega B R$ compensates for the negative space charge without affecting the electric field inside the cylinder.

Now close the switch sw in the circuit of the generator. Conduction electrons escape at the axis into the external circuit, where the $\boldsymbol{v} \times \boldsymbol{B}$ is zero, and return at the rim. The field $\boldsymbol{J}/\sigma$ in the disk points in the direction of the rim.

The case of the motor is similar, except that $\boldsymbol{J}$ points in the $-\hat{\rho}$ direction: current flows in the direction of $\boldsymbol{v} \times \boldsymbol{B}$ in the generator, and in the opposite direction in the motor.

The current in the homoplanar motor depends not only on the characteristics of the motor, but also on the source that feeds it and on its mechanical load, while the current in the generator depends similarly on the applied torque and on the load resistance.

Let us now calculate the potentials V and $\boldsymbol{A}$ for the motor. From Lorrain *et al* (1988, p 349), under steady-state conditions and for a long solenoid,

$$\boldsymbol{A} = (B\rho/2)\hat{\phi} \qquad \partial \boldsymbol{A}/\partial t = 0 \tag{26}$$

$$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\hat{\rho} = \boldsymbol{v} \times \boldsymbol{B} + \frac{\boldsymbol{J}}{\sigma} = \left(-\omega\rho B + \frac{J}{\sigma}\right)\hat{\rho}. \tag{27}$$

Since $J = I/(2\pi\rho s)$, where the current I is independent of the radius,

$$V = \frac{\omega\rho^2 B}{2} - \frac{J\rho}{\sigma}\ln\frac{\rho}{\rho_0}. \tag{28}$$

The quantity ρ_0 is a constant of integration; we have arbitrarily set the second term equal to zero at $\rho = \rho_0$.

In the rotating reference frame, from Lorrain *et al* (1988, pp 287, 303, 321), and from equations (26) and (28),

$$V' = V - vA = \left(\frac{\omega\rho^2 B}{2} - \frac{J\rho}{\sigma}\ln\frac{\rho}{\rho_0}\right) - \omega\rho\frac{B\rho}{2} \tag{29}$$

$$= -\frac{J\rho}{\sigma}\ln\frac{\rho}{\rho_0} \tag{30}$$

$$\boldsymbol{A}' = \boldsymbol{A} - \frac{\omega\rho V}{c^2}\hat{\phi} = \boldsymbol{A} - \frac{\omega\rho}{c^2}\left(\frac{\omega\rho^2 B}{2} - \frac{J\rho}{\sigma}\ln\frac{\rho}{\rho_0}\right)\hat{\phi} \tag{31}$$

$$= \boldsymbol{A} - \frac{\omega^2\rho^2}{c^2}\boldsymbol{A} + \frac{\omega\rho^2 J}{c^2 \sigma}\ln\frac{\rho}{\rho_0}\hat{\phi}. \tag{32}$$

In this last equation the second term on the right is proportional to v^2/c^2 and is therefore negligible. The third term in negligible for the same reason because J is proportional to the sum of two terms, $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$, that are of the same order of magnitude. Thus

$$\boldsymbol{A}' = \boldsymbol{A} \qquad \partial \boldsymbol{A}'/\partial t' = 0 \qquad \boldsymbol{E}' = -\nabla V'. \tag{33}$$

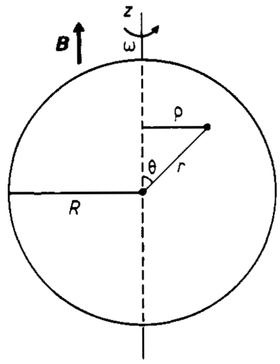


Figure 3.

4. Conducting sphere rotating in a uniform magnetic field

Figure 3 shows a passive conducting sphere of radius R rotating at an angular velocity ω in a uniform magnetic field $\boldsymbol{B}$ parallel to the axis of rotation. The sphere is uncharged. As in section 2.1, there are no induced currents and equations (14) apply. It will be instructive to discuss this simple case (see also Thomson 1893, Mason and Weaver 1929, van Bladel 1984).

We assume that the sphere is rigid: a further paper will discuss the effect of differential rotation with reference to the Earth's core.

We can again disregard the azimuthal current associated with the rotation of the space charge. Then equation (21) applies: the space charge is uniform throughout the sphere, and negative. As we shall see, positive surface charges maintain the net charge equal to zero. This negative space charge establishes a radial electric field

$$\boldsymbol{E}_g = -\frac{\frac{4}{3}\pi r^3(2\varepsilon_r\varepsilon_0\omega B)}{4\pi\varepsilon_r\varepsilon_0 r^2}\hat{\boldsymbol{r}} = -\tfrac{2}{3}\omega B r\hat{\boldsymbol{r}}. \tag{34}$$

The electrostatic field of the *space* charge now cancels the $\boldsymbol{v} \times \boldsymbol{B}$ field only to a certain extent, first because the electric field is along $-\hat{\boldsymbol{r}}$, while the $\boldsymbol{v} \times \boldsymbol{B}$ field is along $\hat{\rho}$, and second because the magnitudes of the two fields are not equal. Equations (14) still apply, however: the combined electrostatic fields of the space and surface charges cancel the $\boldsymbol{v} \times \boldsymbol{B}$ field.

We can find the surface density ς of electrostatic charge as follows. From the Gauss law,

$$\varsigma = \varepsilon_0 E_{r,\text{outside}} - \varepsilon_r \varepsilon_0 E_{r,\text{inside}}. \tag{35}$$

Inside the sphere, in cylindrical coordinates,

$$\boldsymbol{E} = -(\boldsymbol{v} \times \boldsymbol{B}) = -\omega\rho B\hat{\rho} \tag{36}$$

or, in spherical coordinates,

$$\boldsymbol{E} = -\omega r \sin\theta B(\sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}}) \tag{37}$$

$$E_{r,\text{inside}} = -\omega r B \sin^2\theta. \tag{38}$$

Also, inside the sphere, the electric field is radial and the potential along the axis is uniform. Call that potential V_A. Then, on the surface, from equation (36),

$$V_{\text{surface}} = V_A + \omega B\rho^2/2 = V_A + \omega BR^2 \sin^2\theta/2. \quad (39)$$

Outside the sphere, the space charge density is zero, Laplace's equation applies, and we can expand V as a series of Legendre polynomials (Lorrain *et al* 1988, p 225):

$$V_{\text{outside}} = \frac{B_0}{r} + \frac{B_1}{r^2}\cos\theta + \frac{B_2}{r^3}\frac{3\cos^2\theta - 1}{2} + \cdots. \quad (40)$$

The first term, which is the monopole term, implies the existence of a net charge on the sphere. That potential is of no interest; it would apply only if the sphere carried a net charge. The second, dipole, term implies a charge asymmetry between the upper and lower halves of the sphere. This term is also zero because both $\boldsymbol{E}$ and $\boldsymbol{E}_\varrho$ are symmetrical about the equatorial plane. Let us therefore set the third, or quadrupole, term of equation (40) equal to the V of equation (39) at $r = R$:

$$V_A + \tfrac{1}{2}\omega BR^2\sin^2\theta = \frac{B_2}{R^3}\frac{3\cos^2\theta - 1}{2} \quad (41)$$

$$= \frac{B_2}{R^3}(1 - \tfrac{3}{2}\sin^2\theta). \quad (42)$$

Upon matching corresponding terms, we find that

$$B_2 = -\omega BR^5/3 \qquad V_A = -\omega BR^2/3. \quad (43)$$

Thus

$$V_{\text{outside}} = -\frac{\omega BR^5}{3r^3}(1 - \tfrac{3}{2}\sin^2\theta) \quad (44)$$

$$E_{r,\text{outside}} = -\frac{\partial V}{\partial r} = -\frac{\omega BR^5}{r^4}(1 - \tfrac{3}{2}\sin^2\theta). \quad (45)$$

Finally, from equations (35), (38), and (45), the surface charge density is given by

$$\varsigma = \varepsilon_0\omega BR[(\tfrac{3}{2} + \varepsilon_r)\sin^2\theta - 1]. \quad (46)$$

The surface charge density is positive in the region of the equator, and negative near the poles. It is easy to show that the net surface charge is equal to minus the space charge.

The value of the potential inside the sphere follows from equations (36) and (43):

$$V_{\text{inside}} = \omega B\left(\frac{\rho^2}{2} - \frac{R^2}{3}\right). \quad (47)$$

The potential is equal to zero on an axial cylindrical surface whose radius is equal to $(2/3)^{1/2}R$.

5. Conclusion

We have therefore shown that, if a conducting medium moves in a magnetic field, and if the divergence of the $\boldsymbol{v} \times \boldsymbol{B}$ field is not zero, then there exists an electrostatic space charge and its associated electric field. If the conductor is isolated, then surface charges correct for any charge imbalance. If the conductor is isolated and if the magnetic field is axisymmetric, then the field of the electrostatic charges cancels $\boldsymbol{v} \times \boldsymbol{B}$ exactly at every point, there are no induced currents, the rotating conductor does not affect the magnetic field, and the Alfven 'theorem' does not apply.

Acknowledgment

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