THE PHASING OF MAGNETRONS

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by

J. C. Slater

Abstract

The main problems considered in this paper are the phase locking of a magnetron with a small external signal whose frequency is nearly the natural frequency of the magnetron, and the locking together of two or more magnetrons. As a preliminary, we discuss familiar aspects of the operation of a single magnetron into a passive load, including the input impedance of a resonant cavity, the operation of a magnetron into a non-resonant load, the starting of a magnetron, and the operation of a magnetron into a frequency-sensitive load, including a discussion of the stability or instability of modes. Next we come to the main topic, the operation of a magnetron with an external sinusoidal signal. We find that the external signal is equivalent to an external admittance whose phase depends on the phase difference between magnetron and signal. The magnetron will lock to the signal, with such a phase difference that the resulting reactance of the external signal pulls the magnetron frequency to equal the frequency of the signal. If the frequency difference between magnetron and signal is great, and the amplitude of signal small, sufficient frequency pulling cannot occur, and locking does not take place, but the frequency of the magnetron can be modified, and harmonics introduced into its output. If the external signal is replaced by another magnetron, or other magnetrons, there is a similar behavior, and with sufficiently large coupling and sufficiently small frequency differences, coupling will occur, with operation at a weighted mean of the frequencies of the various magnetrons.

THE PHASING OF MAGNETRONS

The problem of operating a number of magnetrons in phase with each other is encountered whenever power greater than that produced by a single magnetron is desired, as in linear accelerators, or in high power radar equipments. This report presents some of the main theoretical aspects of the problem, as they are encountered in the design of the linear accelerator, though it does not treat the specific application to the accelerator. Further experimental work is under way in that project, and no doubt further developments of theory will be indicated as the project progresses. For completeness, this report includes not only information regarding magnetron phasing, but some discussion of magnetron operation in general, and operation into a resonant load.

1. Input Impedance of a Resonant Cavity. — A magnetron cavity is a resonant cavity, provided with an output lead, generally a waveguide. One fundamental property of the magnetron is the input impedance looking in through that waveguide output, as a function of frequency, particularly for frequencies near the resonant frequency of the mode in which it operates. This impedance can be measured by putting a slotted section and standing-wave detector in the output, feeding in a signal from a signal generator, and measuring standing-wave ratio and position of standing-wave minimum as a function of frequency. (See J. C. Slater, "Microwave Electronics", Rev. Mod. Phys., 18, 441 (1946), for this and many other points. We shall give references to this report by the abbreviation ME, followed by chapter and section numbers. Standing-wave ratios are treated in ME, I, 10.) At a frequency considerably removed from resonance the standing-wave ratio will be very high (of the order of 40 db), and the position of standing-wave minimum will vary only slowly with frequency. Interpolating between the positions of standing-wave minima on both sides of resonance, we can get a position which the minimum would have on resonance (we can determine this directly if the magnetron is tunable, by tuning the resonance away from the frequency where we wish to operate, determining standing waves there, and then tuning back so that it resonates at the desired frequency). It is then desirable to use the plane of standing-wave minimum on resonance as a reference plane for measuring the magnetron impedance. Of course, there will be an infinite set of such planes, half a guide wavelength apart; we choose the plane closest to the magnetron, since its frequency variation will be the least. This plane will henceforth be referred to as the plane of the magnetron. A similar plane of reference can be determined for any resonant cavity, in the neighborhood of one of its resonances.

Across the plane of the cavity, the impedance of the cavity as a function of frequency may be approximately written in the form

$$Z = \frac{1/Q_{ext}}{j(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}) + \frac{1}{Q_o}} + Z_1 \qquad (1.1)$$

Here Z is the ratio of the impedance to the characteristic impedance of the guide (as all future impedances and admittances will be, unless otherwise specified). The quantity Q_{ext}, called the external Q, measures the coupling of the cavity to the output line. The resonance frequency is ω_o. The quantity Q_o is the unloaded Q of the cavity. Z_1 is a very small real quantity, measuring the very small impedance leading to the standing-wave ratio of the order of 40 db off resonance (when the first term is zero). We shall neglect Z_1, though for 3 cm and particularly for 1-cm magnetrons, where the losses in the outputs are considerable, it is not entirely negligible. Expression (1.1) represents a series combination of the small resistance Z_1, with a parallel resonant circuit, in which the first term in the denominator is the capacitive susceptance, proportional to the frequency, the second is the inductive susceptance, inversely proportional to frequency, and the third is the resistive conductance, independent of frequency. The fundamental derivation of (1.1) from electromagnetic theory is taken up in ME, III, 5.

By standing-wave measurements we can find the constants of (1.1), as discussed in ME, IV, 2. The value of Z on resonance, neglecting Z_1, is Q_o/Q_{ext}; this equals the standing-wave ratio on resonance. The value of Q_{ext} can be found from the width of the resonance curve, as described in the reference above. The first step in studying a magnetron is to determine the plane of reference, and these fundamental constants of its circuit, by cold test.

If the magnetron is operating, the effect of the electronic discharge will be like that of a non-linear admittance in shunt with the resonant cavity. That is, if g+jb is this admittance (where we shall choose the positive sign to represent the case where the magnetron is delivering power out of the output), and if C is an effective capacity of the magnetron cavity, the input impedance looking into the magnetron is

$$Z = \frac{1/Q_{ext}}{j(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}) + \frac{1}{Q_o} - \frac{g+jb}{C\omega_o}} \qquad (1.2)$$

in which we have neglected Z_1. The meaning of (1.2) must be clearly understood. In the earlier case (1.1), since the magnetron was producing no power, we had to feed a signal from an external signal generator of frequency ω into the cavity, and measured standing-wave ratio and power with that signal, determining impedance from it. In

(1.2), however, the magnetron itself is generating power. It is assumed that it is this power that is being used to observe standing waves and impedance. The negative sign in front of the term in g indicates that the magnetron is a generator rather than a load.

2. Operation of the Magnetron into a Non-Resonant Load. — Clearly when the magnetron is operating, the standing-wave ratio which will be present in the output line will depend on the load. We can exhibit this by writing (1.2) in a different way. Let us assume that the magnetron is operating into a load of admittance $G+jB$ (as before, this represents the ratio of admittance to characteristic admittance of the line). This admittance is to be computed across the reference plane of the magnetron. It may be introduced by a standing-wave introducer, and in the waveguide between load and magnetron we assume a standing-wave detector, to measure the impedance or admittance seen at the reference plane. As in the preceding paragraph, we assume that it is the magnetron's power itself which is used to measure standing waves. Now the quantity Z in (1.2) measures the impedance looking into the magnetron. The impedance looking out of the magnetron, or into the load, across the same plane, will be $-Z$, and its reciprocal, the admittance, will be $-1/Z = G+jB$. Rewriting (1.2), then, we have

$$\frac{g+jb}{C\omega_o} = j\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right) + \frac{1}{Q_o} + \frac{G+jB}{Q_{ext}} \tag{2.1}$$

This is the fundamental equation of magnetron operation, and is discussed in ME, IV, 4 and 5.

To interpret (2.1) we must think more about the characteristics of the electronic discharge. For a given value of d-c current and magnetic field in the magnetron, there will be a functional relation between the r-f voltage on the elements of the magnetron, and the r-f current which flows. This relation is discussed in ME V, 6 and 7. The r-f current of course has two components, one in phase with the voltage, one out of phase. Experiment shows that the component in phase with the voltage decreases with increasing voltage, in a roughly linear manner, the current being finite for very small voltages, but decreasing to zero at a finite voltage. That is, approximately we may write

$$i_{rf} \text{ (in phase)} = \frac{E - V_{rf}}{R} , \tag{2.2}$$

where E, R are constants (see ME IV, 4 for this equation). If this equation is taken as correct, g, which is by definition the ratio of the component of current in phase with the voltage, to the voltage, is

$$g = \frac{(E/R)}{V_{rf}} - (1/R) , \tag{2.3}$$

a hyperbola, becoming infinite for zero r-f voltage, decreasing to zero when $V_{rf} = E$, a finite value. Whether this precise functional relation is assumed or not, the essential point is that g is a definite function of V_{rf}, which can be found by experiment. Furthermore, the power is determined in terms of r-f voltage from these relations; since the power is 1/2 the product of peak voltage and peak component of current in phase with voltage, we have

$$P = \frac{1}{2R} \left(EV_{rf} - V_{rf}^2 \right) , \tag{2.4}$$

a parabola with maximum at $V_{rf} = E/2$, or half the voltage at which g goes to zero. In the more general case in which g is not given by (2.3), we may still assume this same general sort of relationship, with a maximum power for some value of V_{rf}.

The component of current out of phase with the voltage is likewise a function of V_{rf}, as is its ratio to the voltage, which is b. We shall be particularly interested, not in b as a function of voltage directly, but in b as a function of g; for if g is known, V_{rf} can be found from it, and hence b. We find that this relation is approximately linear, with a negative slope: approximately

$$b = b_o - g \tan \alpha, \tag{2.5}$$

where b_o, α are constants. The quantity b_o is difficult to find from experiment, and its value is not well known, but it does not affect our results seriously. The constant α is generally of the order of magnitude of 1/4.

With this understanding of the nature of g and b, we may now return to (2.1). Taking real and imaginary parts of this equation, we have

$$\frac{g}{C\omega_o} = \frac{1}{Q_o} + \frac{G}{Q_{ext}} \tag{2.6}$$

$$\frac{b}{C\omega_o} = \frac{\omega}{\omega_o} - \frac{\omega_o}{\omega} + \frac{B}{Q_{ext}} \sim \frac{2(\omega-\omega_o)}{\omega_o} + \frac{B}{Q_{ext}} , \tag{2.7}$$

where the latter form arises by writing $\omega/\omega_o - \omega_o/\omega = (\omega^2-\omega_o^2)/\omega\omega_o = (\omega-\omega_o)(\omega+\omega_o)/\omega\omega_o$, and setting $\omega = \omega_o$ except in the difference term $\omega-\omega_o$. Now with a given load, the right side of (2.6) is determined. Hence g is determined, and from this the voltage V_{rf} and the power are known. Knowing the r-f voltage, we know b. Then, knowing B from the load, (2.7) determines the frequency of operation.

These relations can be interpreted in a graphical way. We set up an admittance space, in which G is plotted as abscissa, B as ordinate. Then first we plot a line representing the electronic behavior, in which we plot $gQ_{ext}/C\omega_o - Q_{ext}/Q_o$ as abscissa, $bQ_{ext}/C\omega_o$ as ordinate. By (2.5), this is approximately a straight line with a negative slope, making an angle of $-\alpha$ with the axis of abscissas. Next we plot a line whose abscissa is G, and whose ordinate is $B + 2Q_{ext}(\omega-\omega_o)/\omega_o$, the frequency ω being a parameter which varies from point to point of this line. Assuming as we are doing at the moment that G and B are independent of frequency, this is a vertical straight line. The intersection of our two lines then, by (2.6) and (2.7), determines the operation, its abscissa determining g and hence the r-f voltage and power, and the ordinate determining frequency.

We can see from (3.7) that the frequency is affected by two things besides the resonant frequency of the cavity. First, if b changes, but B stays the same, the frequency will change. This is the phenomenon of frequency pushing. As the d-c conditions of operation change, the values of g and b as functions of r-f voltage change. It is found that the effect on the relation (2.5) between b and g is a change in b_o, or a vertical displacement of the curve, without much change in α. Thus by (2.7) there is an effect on frequency. The other effect is that of a change in B, the reactance of the load. This effect on the frequency is called frequency pulling, and from (2.7) we see that the amount of frequency pulling is inversely proportional to Q_{ext}. In fact, the pulling figure of a magnetron is defined as the extreme change of frequency when the reflection coefficient of the load goes around a a circle corresponding to a standing wave of 1.5. In our G-B space, this corresponds to a circle extending from G = 2/3 to G = 3/2, and having thus a diameter of 3/2 - 2/3 = 5/6. The extreme variation of frequency produced by any admittance on this circle then corresponds to the amount of vertical displacement possible without losing an intersection between the circle and the line representing the relation between g and b. As in Figure 1,

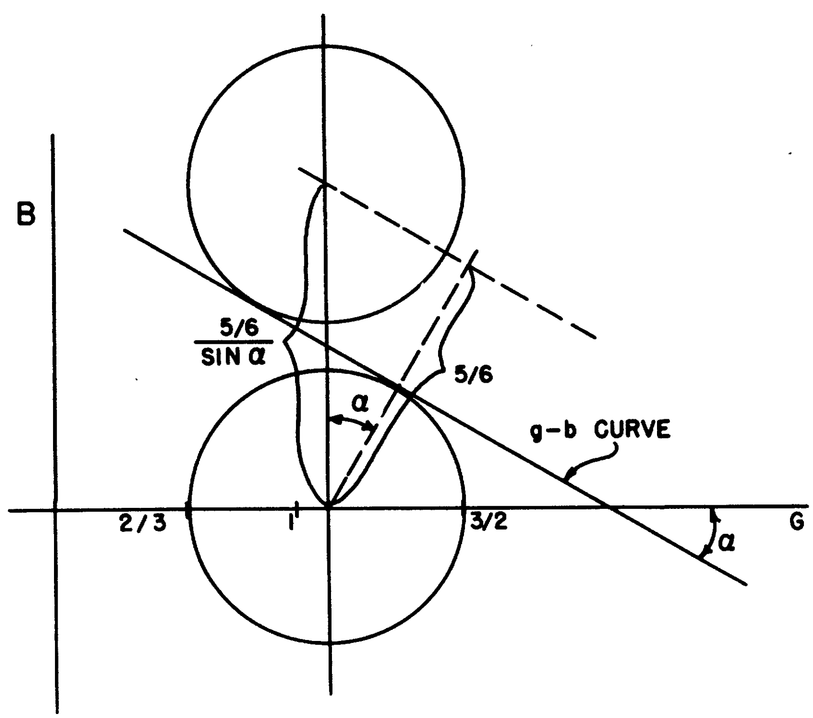

Figure 1

we see that this vertical displacement is related to the diameter of the circle by the relation, vertical displacement = $(5/6)/\sin\alpha$. Thus from (2.7), where we find that the vertical displacement equals twice the change in frequency divided by the average resonance frequency, we have

$$\text{Pulling figure} = \frac{5}{12 \sin\alpha} \frac{\omega_o}{Q_{ext}} \quad . \tag{2.8}$$

In the experimental study of magnetron operation, we adjust the output load, or G+jB, and measure the output power and the frequency of operation. We can then plot contours of constant power, and of constant frequency, on the G-B plane. From what we have just seen, the contours of constant power should be vertical lines corresponding to G = constant, and the contours of constant frequency should be a set of lines sloping downward with angle α to the axis of abscissas, the contour for $\omega = \omega_o$ being, from (2.5) and (2.7),

$$B = \left[b_o \frac{Q_{ext}}{C\omega_o} - \frac{Q_{ext}}{Q_o} \tan\alpha \right] - G \tan\alpha \tag{2.9}$$

and the contours for other frequencies being displaced upward by an amount $2Q_{ext}(\omega-\omega_o)/\omega_o$. It is usually found experimentally that this vertical spacing of the frequency contours is in good agreement with the value as predicted from the Q_{ext} as determined by cold test; and that the power contours actually are vertical lines provided G is neither excessively large or excessively small; in these limits, other complicating features come in, which we shall not bother with at present.

We have spoken as if the values of g and b were known to start with. This of course is not the case. It is rather through observation of the operating characteristics that these quantities are known. Thus an observation of frequency contours clearly gives g as a function of b, using (2.6) and (2.7), and it is from such experiments that we deduce the linear form (2.9) which approximately represents observation. Similarly we can get the relation between g and V_{rf} from the power observation. The power P is $\frac{1}{2} g V_{rf}^2$. If we observe the power, and find $g/C\omega_o$ from (2.6), from the measured G and the values of Q_o and Q_{ext} found from cold test, and if we can estimate C (which we can do by study of the internal circuit of the magnetron, in a way suggested in ME III,6), then we can get g. Hence from the power we find V_{rf}. It is in this way that the approximate equations (2.2) and (2.3) have been set up, to describe the results of such experiments.

In determining the power, one caution is necessary. Not all the power produced by the oscillator finds its way to the load; some of it is absorbed in losses inside the magnetron. To find the fraction of power produced which is delivered to the load, which is called the circuit efficiency, we use Eq. (2.1). We note that it is like a shunt combination of capacity, inductance, loss conductance proportional to $1/Q_o$, and load conductance proportional to G/Q_{ext}, as well as load susceptance. Since in a parallel circuit there is the same voltage across all elements of the circuit, the power dissipated in the magnetron loss will be to the power dissipated in the load as $1/Q_o$ is to G/Q_{ext}. Thus we shall have

$$\eta_c = \frac{G}{G + Q_{ext}/Q_o} . \tag{2.10}$$

We must in every case divide the observed power by the circuit efficiency, in order to determine the power produced by the magnetron, which we must use in finding the relation between g and V_{rf}.

We have plotted the results of observation in a G-B plane, and this plot is the most useful for theoretical interpretation. However, in practice, it is common to plot in a Smith chart, for convenience in making transformations from one point of the line to another. Such a plot is generally called a Rieke diagram, though Rieke himself has used the plot in the G-B plane more than in the Smith chart. We shall not go further with the appearance of the Rieke diagram, but shall assume in all cases that the results of observation are to be interpreted in a G-B plane. In doing this, it is essential that the plane of reference be that described at the beginning of this section. The nature of the Rieke diagram in the Smith chart is discussed in ME IV,5, the Smith chart or reflection coefficient plane being taken up in ME, I.

3. The Starting of a Magnetron. — In the preceding section we have considered the steady state operation of a magnetron. In starting, the situation is quite different. The problem is discussed in ME IV, 6, and we reproduce the discussion given there with little change. For a short time interval during the build-up, we may assume that the amplitude is increasing exponentially with the time, so that formally we may treat the frequency as being complex, the imaginary term representing the exponential increase. Thus if $\omega = \omega_1 + j\omega_2$, the time variation of voltage will be according to $e^{-\omega_2 t}\, e^{j\omega_1 t}$, so that $-\omega_2 = d \ln V_{rf}/dt$, where V_{rf} is the voltage amplitude. Substituting a complex frequency in (2.1), we have instead of (2.6),

$$\frac{g}{C\omega_o} = \frac{1}{Q_o} + \frac{G}{Q_{ext}} - \frac{2\omega_2}{\omega_o} = \frac{1}{Q_o} + \frac{G}{Q_{ext}} + \frac{2}{\omega_o}\frac{d}{dt}\ln V_{rf} . \tag{3.1}$$

Eq. (2.7) remains unchanged. Thus we have a differential equation for the time variation of V_{rf}, if we know g as a function of V_{rf}, as for example from (2.3). If we assume that value, the differential equation can be integrated, giving

$$V_{rf} = \frac{E}{RC\omega_o}\ \frac{1}{(1/RC\omega_o + 1/Q_L)}\ \left\{1 - e^{-\frac{\omega_o}{2}\left(\frac{1}{RC\omega_o} + \frac{1}{Q_L}\right)t}\right\} \tag{3.2}$$

where

$$\frac{1}{Q_L} = \frac{1}{Q_o} + \frac{G}{Q_{ext}} .$$

In other words, from (3.2), we see that the voltage in the magnetron builds up exponentially, the time constant being determined from the loaded Q, Q_L, with an additional loading term $1/RC\omega_o$. Since the loaded Q is ordinarily small (say 100 for a 10-cm magnetron), this indicates that it does not take many cycles for the magnetron to build up the voltage in the cavity, and get into full operation. In our plot in the G-B plane, we note that during the build-up we are on the b-g curve, but not at the point corresponding to the G and B of the load. Instead, we start with low voltage, or large values of g, or far to the right in this plot, and gradually move to the left along the g-b curve, until we come to the point corresponding to stable operation.

4. <u>Operation of a Magnetron into a Frequency-Sensitive Load</u>. — We have been considering the operation of a magnetron into a load which was independent of frequency. On the other hand, in many cases we wish to operate into a circuit, such for example as a resonant cavity, which has an admittance depending on frequency. In this case the quantity on the right side of Eq. (2.1) becomes a more complicated function of frequency than we have so far considered. We can still handle the problem, however, by the same fundamental principles we have been using, and at the same time can throw more light on the nature of those principles.

Let us rewrite (2.1) in the form

$$(g+jb)\frac{Q_{ext}}{C\omega_o} = y(\omega) = 2jQ_{ext}\frac{(\omega-\omega_o)}{\omega_o} + \frac{Q_{ext}}{Q_o} + G(\omega) + jB(\omega). \qquad (4.1)$$

The function $y(\omega)$ is a complex function of the variable ω. In steady state operation we are concerned only with real values of the frequency, but in the preceding section we see that we need to consider complex frequencies as well. We may then consider the mapping of the ω plane onto the y plane, as in studying functions of a complex variable. We may draw the contours corresponding to ω_1 = constant, ω_2 = constant, in the y plane. Since y is an analytic function of ω, the transformation will be conformal, and squares in the ω plane will transform into approximate squares in the y plane. Thus we may have a set of contours as in Figure 2. We have drawn lines to correspond to equally spaced values of the real and imaginary parts of the frequency, and have arranged to have the real part of frequency increasing in general upward, and the imaginary part increasing in general to the left, which is the usual situation. We may now draw on the same plot the g-b line, plotting $bQ_{ext}/C\omega_o$ as a function of $gQ_{ext}/C\omega_o$. Then the magnetron will start to build up its oscillation at a low value of r-f voltage, or a large value of g. Here it will find a large negative value of ω_2, or will have a high rate of build-up. Its voltage will increase, its g will decrease, or the point representing it will move to the left along the g-b curve, as shown by the arrow in the figure. At the point P where the g-b curve intersects the curve $\omega_2 = 0$ it will come to equilibrium, oscillating with the frequency given by the value of ω_1 appropriate to this point.

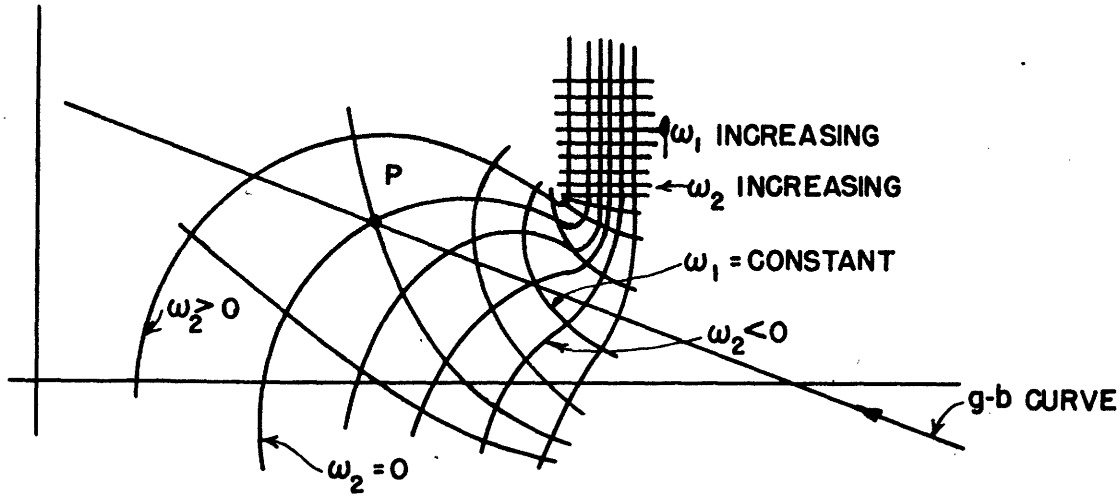

Figure 2

The equilibrium in the case shown above is a stable equilibrium: an increase of g, or decrease of voltage, brings us to a region of negative ω_2, or of increasing voltage, while an increase of voltage carries us to a positive ω_2, or decreasing voltage. Clearly we can also have states of unstable equilibrium, in which a small change of voltage in either direction leads to a further change of voltage. In the next figure (Fig. 3) we show the curve of $\omega_2 = 0$ for a case in which the g-b curve intersects in three places rather than one. In the intersections marked P_1, P_3, we have stability, while in P_2 we have instability: a displacement from P_2 in the direction of P_1 will result in a transfer to P_1, while a displacement toward P_3 will end up at P_3.

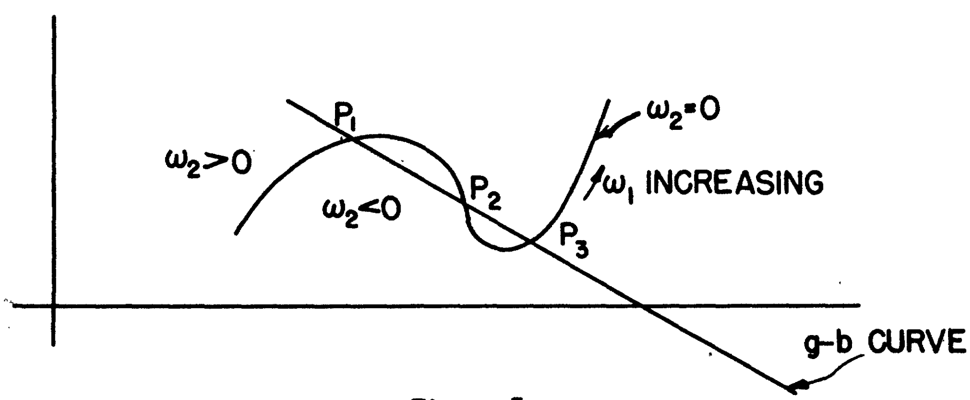

Figure 3

In a case as shown above, it is true that both P_1 and P_3 represent stable operating points. However, if the oscillator were started in the circumstances shown, it is clear from our preceding discussion that the point representing the oscillation will start at the right on the g-b curve, and will move to the left as the voltage builds up. Since it will meet point P_3 first, this will be the state of oscillation which will actually be set up. However, suppose that we have some control over the circuit, say by a tuning knob, so that we can shift the curve corresponding to $\omega_2 = 0$ bodily upward, or otherwise distort it. (This can always be accomplished, in a tunable magnetron, by tuning it, for this changes ω_o in (4.1), and as we can see that shifts the whole function y upwards). Then we can cause points P_2 and P_3 to approach each other, and can remove the intersection with the g-b curve represented by these points. When this happens, the magnetron would find itself no longer in a steady state, and its ω_2 would again be less than zero, so that the voltage would proceed to build up more. The point would move further to the left on the g-b curve, and would end up at a point near P_1. If the magnetron were in continuous operation, we could then tune it back, and would proceed to an equilibrium manner to point P_1. This could naturally not be done in a pulsed magnetron, since each pulse represents a new start, and the pulses are so fast that it is not practicable to tune the load during a pulse.

5. Operation of a Magnetron with an External Sinusoidal Signal. — Until now, we have considered the operation of a magnetron into a passive load. Now we come to the first case of the phasing of magnetrons. The essential requirement for phasing of an oscillator is a load whose admittance depends on the oscillator's phase. Such a load is provided if there is an external signal being fed into the oscillator through the output. Let us see how this comes about.

Suppose the current and voltage being delivered by the magnetron, at its plane of reference, are $i\, e^{j\omega t}$ and $V\, e^{j\omega t}$, and let there be an additional signal of frequency ω_1, fed in from outside, so that its current and voltage at the magnetron's plane of reference are $i_1 e^{j\omega_1 t}$ and $V_1 e^{j\omega_1 t}$. We note that if, for instance, i_1 is in phase with i, V_1 will be out of phase with V, in order that the direction of power flow in the magnetron power and the external signal may be opposite to each other. Now let us superpose these two fields, and divide current by voltage, to find the admittance. This will be

$$\frac{i\, e^{j\omega t} + i_1 e^{j\omega_1 t}}{V\, e^{j\omega t} + V_1 e^{j\omega_1 t}} = \frac{i}{V}\, \frac{\left[1 + (i_1/i) e^{j(\omega_1-\omega)t}\right]}{\left[1 + (V_1/V) e^{j(\omega_1-\omega)t}\right]}\,. \tag{5.1}$$

In other words, there is a time-dependent term varying with frequency $\omega_1-\omega$ in the admittance. If i_1/i and V_1/V are small, we may write this admittance as

$$G + jB + a\, e^{j(\omega_1-\omega)t}, \tag{5.2}$$

where

$$G + jB = i/V,\quad a = (G+jB)\left(\frac{i_1}{i} - \frac{V_1}{V}\right). \tag{5.3}$$

Thus the ratio of a, the amplitude of the time-dependent term, to the constant admittance, depends on the ratio of current or voltage in the incoming signal to that sent out by the magnetron. We recall, as was mentioned earlier, that the two terms i_1/i and $-V_1/V$ will have the same sign. Even if a is not small, we still note from (5.1) that the admittance is a bilinear function of the complex quantity $e^{j(\omega_1-\omega)t}$. As this quantity goes around a circle in the complex plane, which it does as time increases, the corresponding point in the admittance plane must then also go around a circle. It will obviously rotate periodically, with the same period $\omega_1-\omega$ as before, but it will no longer rotate uniformly, so that it can be described by a Fourier series. The expression corresponding to (5.2), then, for large amplitudes, will contain harmonics of $\omega_1-\omega$, as well as the fundamental. For our purposes we may neglect these harmonics, and assume an expression of the form (5.2) in all cases.

Under the action of this time-dependent admittance of the load, there will be a frequency-pulling phenomenon in the magnetron. This phenomenon has been described, in very general terms, by Huntoon and Weiss (Synchronization of Oscillators, Technical Report of National Bureau of Standards, Division XIII, Ordnance Development Division, Section 6-Electronics, Radiation Group, July 24, 1946). Since we shall be concerned with the phase of the oscillator, as well as its frequency, let us assume that its phase is ϕ, so that its frequency of operation ω equals $\dot{\phi}$, and we may replace ωt by ϕ. Then the admittance (5.2) becomes $G + jB + a\, e^{j(\omega_1 t-\phi)}$. In this form it is obvious that the admittance depends on the phase of the magnetron, and it is natural that a very tight locking of the phase is possible.

Before proceeding with the mathematical discussion, we can give a qualitative discussion of the process of phasing which proves on analysis to be correct. We start from our fundamental equation (2.1), but modified to include the phase-dependent load which we are now considering. That is, we have

$$\frac{g+jb}{C\omega_o} = j\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right) + \frac{1}{Q_o} + \frac{G + jB + a\, e^{j(\omega_1 t - \phi)}}{Q_{ext}} . \tag{5.4}$$

The time-dependent term in (5.4) may be separated, in the usual way, into real and imaginary parts, each of which depends on the phase difference between magnetron and external signal. If we let this phase difference be θ, defined by

$$\theta = \phi - \omega_1 t, \tag{5.5}$$

and if for convenience we assume that a is real, these real and imaginary parts of (5.4) are

$$\frac{g}{C\omega_o} = \frac{1}{Q_o} + \frac{G}{Q_{ext}} + \frac{a\cos\theta}{Q_{ext}} , \tag{5.6}$$

$$\frac{b}{C\omega_o} = \frac{2(\omega-\omega_o)}{\omega_o} + \frac{B}{Q_{ext}} - \frac{a\sin\theta}{Q_{ext}} . \tag{5.7}$$

We now consider (5.7). After solving for frequency ω, it indicates a frequency differing from the value we should have without the incoming signal, by the amount $\frac{\omega_o}{2Q_{ext}}$ a sin θ, provided θ can be treated as a constant. But by (5.5), θ can be constant only if $\dot{\phi} = \omega_1$, or if $\omega = \omega_1$. That is, this can be true only if the magnetron is locking to the external signal. In this case, we may solve (5.7) for sin θ, finding

$$\frac{a \sin \theta}{Q_{ext}} = \frac{2(\omega_1 - \omega_o)}{\omega_o} + \frac{B}{Q_{ext}} - \frac{b}{C\omega_o} = \frac{2(\omega_1 - \omega')}{\omega_o} \tag{5.8}$$

where ω' is the frequency with which the magnetron would operate with the existing B (frequency pulling) and b (frequency pushing) in the absence of the external signal. Equation (5.8) suggests that there will be a solution in which the magnetron locks to the signal, with a definite phase difference θ, which reduces to zero when the external signal has exactly the frequency with which the magnetron would operate without the signal. The maximum value which sin θ can have is of course ± 1; and this fixes extreme limits for $\omega_1 - \omega'$, the frequency difference between the signal and the natural frequency of magnetron operation, beyond which locking is impossible. For a constant value of θ, (5.6) then indicates that the external signal will have the effect of adding a resistive as well as a reactive term to the load, and this, by the principles discussed in earlier sections, will result in a modified r-f voltage and power. With only sin θ known from (5.7) or (5.8), the sign of cos θ is ambiguous, but we shall show later that the sign is such that when the magnetron is locked to the signal, a cos θ is negative, so that the signal acts like a negative conductance, or the negative sign represents a power flow from the signal into the magnetron. This term is maximum when $\omega_1 = \omega'$, and reduces to zero when $\omega_1 - \omega'$ is at its extreme limit.

The condition of locking in, which we have described, is a steady state, but it is obviously not the general solution of the problem. In the next section we shall set up a differential equation, following Huntoon and Weiss, which represents the process of approach to this steady state. We shall find that if the external signal is suddenly impressed, the phase will rapidly adjust itself to the proper value (5.8), with a time constant substantially the same as that by which the r-f voltage adjusts to its final value, as discussed in Sec. 3. On the other hand, if the signal frequency is too far from the magnetron frequency for locking to occur, we shall show that the signal perturbs the magnetron operation, in that it tends to pull the magnetron frequency toward the signal frequency, and at the same time introduces harmonics into the magnetron operation, both phenomena becoming large as the signal approaches the limiting frequency at which it can produce locking.

6. Analysis of the Process of Locking. – To set up a differential equation describing the process of locking, let us start with (5.7), introducing the frequency ω' as in (5.8), and let us replace ω, which equals $\dot{\phi}$, by $\dot{\theta} + \omega_1$, from (5.5). Then we have

$$\frac{d\theta}{dt} = \frac{a\omega_o \sin\theta}{2Q_{ext}} + (\omega' - \omega_1). \tag{6.1}$$

This differential equation may be solved at once, by multiplying by dt, dividing by the right-hand side, integrating, and solving the resulting equation. The answer is

$$\tan\frac{\theta}{2} = \frac{a\omega_o}{2(\omega_1-\omega')Q_{ext}} - \sqrt{\left(\frac{a\omega_o}{2(\omega_1-\omega')Q_{ext}}\right)^2 - 1}\ \frac{1+e^{A(t-t_o)}}{1-e^{A(t-t_o)}} \tag{6.2}$$

where

$$A = \sqrt{\left(\frac{a\omega_o}{2Q_{ext}}\right)^2 - (\omega_1-\omega')^2}\ .$$

Here t_o is the constant of integration. This differential equation (6.1), and its solution are discussed by Huntoon and Weiss and its application to the phasing of triode circuits was discussed by Adler, Proc. I.R.E., 34, 351 (1946).

There are two cases of this solution, depending on whether the magnitude of $\omega_1-\omega'$ is greater than or less than $a\omega_o/2Q_{ext}$. From our previous discussion, particularly of Eq. (5.8), we see that the case where $\omega_1-\omega'$ is less than this quantity is the case where the magnetron locks to the signal, and the case where it is greater is that when the magnetron does not lock in. Let us first consider the case of locking.

For the case of locking in, A is real. Then as time goes on, the exponential in (6.2) becomes infinite, and tan $\theta/2$ approaches a limiting value determined from (6.2). By a little trigonometric manipulation, we can show that the corresponding value of θ is

$$\theta = \pi - \sin^{-1}\frac{2(\omega_1-\omega')Q_{ext}}{a\omega_o} \tag{6.3}$$

where in (6.3) we are to use that particular value of the inverse sine which goes to zero when its argument goes to zero. Thus in this limit sin θ approaches the value $2(\omega_1-\omega')Q_{ext}/a\omega_o$, the value derived in (5.8) for the limiting value. In other words, since θ approaches a constant value, the magnetron locks in, as described earlier. Now, however, we have determined the time constant A with which this locking in is accomplished. At the same time we find from (6.3) that the value of θ which corresponds to locking in is that which approaches the magnetron frequency ω', so that the term a cos θ in (5.6) approaches a negative value, indicating power flowing from the signal into the magnetron, as we have previously mentioned.

If the external signal is suddenly applied, as we have seen, the phase of the magnetron will gradually pull in to its limiting value, and it is easy to see from (6.2) that θ will never vary by more than π in the process. In other words, the locking in in frequency occurs immediately, but the phase gradually adjusts itself. At the same time, of course, according to (5.6), the r-f voltage and power output must adjust themselves. We may compare the time constant by which this power adjustment will occur with that for phase, by comparing Eqs. (5.6) and (3.1). At the instant when the external signal is applied, g will find itself out of adjustment for the new load. Taking for simplicity the case where $\cos\theta = -1$, we see that $-a/Q_{ext}$ is to be identified with $(2/\omega_o)d \ln V_{rf}/dt$. That is, the time constant for approach of r-f voltage to equilibrium in this case is $a\omega_o/2Q_{ext}$, which by (6.2) is the same as the value of the time constant A for approach of phase to equilibrium. We note from (6.2) that as $\omega_1-\omega'$ approaches $a\omega_o/2Q_{ext}$, or as we approach the edge of the frequency band over which lock-in is possible, the time constant for locking in of phase becomes larger, or A becomes smaller. Similarly as $\cos\theta$ becomes less than unity, the time constant for adjustment of voltage becomes larger.

Next we consider the second case, that where $\omega_1-\omega'$ is greater than $a\omega_o/2Q_{ext}$, so that locking in never occurs. In that case A becomes imaginary, and (6.2) can be written in the form

$$\tan\frac{\theta}{2} = \frac{a\omega_o}{2(\omega_1-\omega')Q_{ext}} + \sqrt{1 - \left(\frac{a\omega_o}{2(\omega_1-\omega')Q_{ext}}\right)^2} \cot\frac{1}{2}A'(t-t_o) \quad (6.4)$$

where $A' = A\sqrt{-1}$. For large values of $\omega_1-\omega'$, this approaches $\tan\theta/2 = \cot A'(t-t_o)/2$, so that $\theta = \pi - A'(t-t_o) = \pi - (\omega_1-\omega')(t-t_o)$. Then from (5.6) we have $\dot{\phi} = \omega'$, so that the frequency of the magnetron is unaffected by the external signal. For smaller values of $\omega_1-\omega'$, however, the situation will be different. When $A'(t-t_o)$ increases by 2π, θ will still decrease by 2π, so that we shall still have the average time rate of change of θ equal to $-A'$, but θ will no longer be a linear function of time. It will instead be a linear function, with a periodic function of period $2\pi/A'$ superposed on it. Thus in the first place, considering the value of A', we shall have

$$\dot{\phi} = \omega_1 - \sqrt{(\omega_1-\omega')^2 - \left(\frac{a\omega_o}{2Q_{ext}}\right)^2} \quad (6.5)$$

as the average frequency of the magnetron. This is a value which equals the frequency of the external signal at the edge of the lock-in band, but which gradually reduces to the unperturbed frequency ω' of the magnetron as the signal is tuned far from the magnetron's frequency. At the same time, on account of the periodic variation of θ with time, we shall have essentially a frequency modulation in the magnetron's output, with side bands whose frequencies are integral multiples of $A'/2\pi$. This quantity is the frequency difference between the magnetron and the external signal.

We may expect, then, the following situation as a magnetron is operated continuously, and a continuous external signal is tuned closer and closer to the magnetron's frequency. As the external signal approaches the frequency for which lock-in is possible (a frequency which is further from the magnetron's normal frequency, the greater the amplitude of the signal, or the greater the frequency pulling of the magnetron), the magnetron's frequency will pull toward that of the external signal, and at the same time side bands will build up, separated from the magnetron frequency by integral multiples of the frequency separation between the magnetron and signal frequencies. These side bands will increase in intensity as the external signal comes closer to locking the magnetron. Finally lock-in will occur, the side-bands will disappear, and the magnetron will operate in the same frequency as the external signal. As the signal is tuned to the natural operating frequency of the magnetron, the magnetron will follow its frequency, but will be changing its phase with respect to the external signal. As the external signal is tuned away from the magnetron resonance on the other side, the same events will be observed in reverse sequence.

The writer has not carried out such experiments with c-w magnetrons, but has performed an experiment with two reflex klystrons coupled to each other through an attenuator, the spectra of the two being observed by a probe in the line joining the two klystrons. As one of the klystrons was tuned toward the other, phenomena just like those of the preceding paragraph were observed, the two frequencies of the oscillators as observed on a spectrum analyzer pulling together, very conspicuous harmonics building up, and finally the two oscillators locking together, and tuning together with the tuning adjustment of either oscillator. The theory of this section was not available at the time these observations were made, so that no test was made of the numerical aspects of the theory. However, one observation was made which bears out the correctness of the theory. We should expect that if the frequency difference between the incoming signal and the oscillator remained fixed, and if the strength of the incoming signal was increased by decreasing the attenuation between the two oscillators, we should approach the condition of lock-in, just as if we brought the two frequencies together. This was observed in a very conspicuous manner. With the two oscillators tuned some distance apart, and a considerable attenuation between them, the attenuation was then decreased. The harmonics appeared, the fundamental frequencies of the two oscillators pulled together and they finally locked.

The phenomena we have been describing in the preceding paragraphs refer to continuous operation of magnetrons or other oscillators. With pulsed operation we naturally expect the situation to be somewhat different. As the magnetron is started, if the external signal is already present, we may expect the oscillation to build up in voltage, and to stabilize in phase, much as in the preceding discussion. We must remember, however, that the spectrum as observed will be complicated by the finite length of the pulse. The build-up time as derived from our discussion is generally considerably shorter than the ordinary pulse length, so that the breadth of the band in the spectrum representing the line will be smaller than the frequency difference between external signal and magnetron. This would not always be the case, however, and a more complete discussion than given here would be necessary to describe the appearance of the spectrum in all cases.

7. Operation of Two Magnetrons into a Single Load. — In our discussion of the preceding section, we have assumed the external signal to be unaffected by the behavior

of the magnetron which it was phasing. This of course would generally not be the case. If the external signal were produced by another oscillator of roughly the same power output as the magnetron, acting through an attenuator to reduce its strength at the magnetron, then the magnetron under discussion can equally well send power through the attenuator to the oscillator producing the signal, and can react on that. We must expect then that in general each of two or of several oscillators coupled together will affect the others, in a symmetrical manner. We shall find, however, that this does not affect the results in an important way.

As a first indication of this, let us consider the two oscillators, coupled together. Let the phase of one be ϕ_1, and of the other ϕ_2. The admittance of each will then contain a term depending on the phase difference $\phi_2-\phi_1$. Proceeding as in (5.4), (5.8), and (6.1), we have

$$\frac{d\phi_1}{dt} = \omega_1' + \frac{a_1\omega_1'}{2Q_{ext,1}} \sin(\phi_1-\phi_2)$$

$$\frac{d\phi_2}{dt} = \omega_2' + \frac{a_2\omega_2'}{2Q_{ext,2}} \sin(\phi_2-\phi_1). \tag{7.1}$$

These must be solved simultaneously. Subtracting the first from the second, we have an equation for $\phi_2-\phi_1$:

$$\frac{d(\phi_2-\phi_1)}{dt} = \left(\frac{a_2\omega_2'}{2Q_{ext,2}} + \frac{a_1\omega_1'}{2Q_{ext,1}}\right) \sin(\phi_2-\phi_1) + (\omega_2'-\omega_1'). \tag{7.2}$$

This is an equation of the same form as (6.1), so the same discussion applies here as there. If the resonant frequencies ω_1' and ω_2' of the two magnetrons are not too far apart, they will lock in, the phase difference between them settling down to a constant value. Then (7.1) will give the asymptotic values of $\dot{\phi}_1$ and $\dot{\phi}_2$, the frequencies of the two magnetrons, which must equal each other for lock-in. These equations, or the limiting form of (7.2), will also give the final value of phase difference. From (7.1), we see that this will be such that its effect on each magnetron will be to introduce a reactive load sufficient to pull its frequency to the operating frequency, which will not equal the frequency of either magnetron. Rather, if the two magnetrons have identical properties ($a_1 = a_2$, $Q_{ext,1} = Q_{ext,2}$), but are tuned to slightly different frequencies, the operating frequency, from (7.1), will be the average of the two frequencies, so that if either one is tuned, the operating frequency will tune by one-half the amount by which this one is tuned.

This last result suggests a different approach to the problem of coupled magnetrons, which is legitimate and simple. Suppose we have two magnetrons coupled through a matched T to a load in the following manner. With a suitably matched T, we can find reference planes in each of the three branches, such that the sum of the admittances looking out from the T into each of the three outputs equals zero; that is, the admittance looking into any output equals the sum of the admittances looking out the other two, or the admittances appear to be in shunt. Now let magnetrons be attached to two of these outputs, with such line lengths that the magnetron planes of reference are at the reference planes of the two branches, and let a load be inserted in the third branch, its value being measured across the plane of reference in that branch. Then by using (1.2) for the impedance looking into a magnetron, and its reciprocal for the admittance, and letting the load admittance be $G + jB$, the relation for the shunt T is

$$- (G+jB) = Q_{ext,1}\left\{j\left(\frac{\omega}{\omega_1} - \frac{\omega_1}{\omega}\right) + \frac{1}{Q_1} - \frac{g_1+jb_1}{C_1\omega_1}\right\}$$

$$+ Q_{ext,2}\left\{j\left(\frac{\omega}{\omega_2} - \frac{\omega_2}{\omega}\right) + \frac{1}{Q_2} - \frac{g_2+jb_2}{C_2\omega_2}\right\}. \qquad (7.3)$$

Here the external Q, unloaded Q, etc., of the two magnetrons are denoted by subscripts 1 and 2, respectively. We thus see that the magnetron circuits, and the electronic conductances, of the two magnetrons are effectively in shunt with each other under these circumstances. We can then proceed as in Sec. 2 to find the frequency of operation, and r-f voltage. Suppose B, b_1, b_2 are zero, or that they are taken care of by replacing ω_1, ω_2 by ω_1', ω_2', as before. Then taking the imaginary part of (7.3), we have

$$0 = Q_{ext,1}(\omega-\omega_1') + Q_{ext,2}(\omega-\omega_2'). \qquad (7.4)$$

We note that this equation is satisfied by the values from the two simultaneous equations (7.1), provided we set $\omega = \dot{\phi}_1 = \dot{\phi}_2$ in that equation, as we have if the magnetrons are locked in, and if we assume $a_1 = a_2$, and that the frequency difference between ω_1' and ω_2' is small. Thus this simple concept of the magnetrons' being in shunt leads to the same result for the final frequency of operation as our previous method, based on the differential equation (7.2). This final frequency, found by solving (7.4), is

$$\omega = \frac{\omega_1' Q_{ext,1} + \omega_2' Q_{ext,2}}{Q_{ext,1} + Q_{ext,2}}, \qquad (7.5)$$

a weighted mean of the frequencies of the two magnetrons, weighted in proportion to their external Q's; that is, a magnetron with high external Q, or low pulling figure, tends to fix the final frequency more than one with low external Q, or high pulling figure.

We can look at Eq. (7.3) in another way. Let the admittance

$$Q_{ext,2}\{j(\omega/\omega_2-\omega_2/\omega) + 1/Q_2 - (g_2+jb_2)/C_2\omega_2\}$$

which we see looking into the second magnetron across its plane of reference be called G_2+jB_2, where of course for the operating magnetron G_2 will be negative. Then we can write (7.3) in the form

$$\frac{g_1+jb_1}{C_1\omega_1} = j\left(\frac{\omega}{\omega_1} - \frac{\omega_1}{\omega}\right) + \frac{1}{Q_1} + \frac{(G+jB + G_2 + jB_2)}{Q_{ext,1}} \quad . \qquad (7.6)$$

If the phase of magnetron 1 is ϕ_1, that of magnetron 2, ϕ_2, then the admittance G_2+jB_2 as interpreted by the first magnetron will contain a phase factor $e^{j(\phi_2-\phi_1)}$, as before, so that (7.6) is equivalent to (5.4). Furthermore, we can find the value of a by considering the properties of magnetron 2. For instance, suppose each magnetron is designed to operate into a matched load. Then each of the terms on the right side of (7.3) will equal -1, when the load is correctly chosen. To have the two magnetrons operating into proper loads when connected by the T, we must clearly have the load (G+jB) of (7.3) equal to 2, or have twice the admittance of a matched load. If then the magnetrons are operating in this manner, the quantity G_2+jB_2 as seen looking into magnetron 2 will be -1, so that by comparing (7.6) and (5.4) we see that in this case a = 1 (or -1, which amounts to the same thing, since we have a phase at our disposal in the angle θ). This allows us to use our analysis of Sections 5 and 6, and to conclude that the maximum frequency difference with which locking is possible, with this arrangement of magnetrons and load, is $\omega_1/2Q_{ext,1} + \omega_2/2Q_{ext,2}$. This means, as far as order of magnitude is concerned, that if the resonance curves of the two magnetrons, as determined by their loaded Q's, overlap in frequency, they will lock in with each other. This of course is a tighter locking then we often find, since if there is a decoupling between the magnetrons, by an attenuator or other means, which we do not have with this case of the matched T coupling, the constants a_1 and a_2 of (7.2) may be much less than unity, and the corresponding limit of locking-in may be much smaller.

8. Operation of N Magnetrons into a Single Load. — We can now generalize the results of the preceding section to the operation of N magnetrons into a single load. Let us suppose we can set up a circuit, possessing the same properties as the matched T in our preceding case; that is, a cavity possessing N+1 outputs, such that the sum of the admittances locking out the N+1 outputs equals zero. We then have an equation similar to (7.3), if N magnetrons are attached to N of the outputs, and an admittance G+jB to the other, except that there are N terms on the right side of the equation. If each magnetron is designed to operate into a matched load, we then wish to make G = N for best operation. Since it is difficult to make a load with this large admittance, it is more convenient in practice to have N outputs for magnetrons, and N for loads, all effectively in shunt with each other, and to put matched loads at each of the load outputs. A practical method of realizing such a circuit has been described by Bostick, Everhart, and Labitt, Technical Report No. 14 of Research Laboratory of Electronics, September 17, 1946.

Setting up our circuit as we have just described, we may then write an equation like (7.6) but differing from it in that there will be magnetron admittances from 2 to N on the right-hand side. We really should handle this problem by simultaneous equations, as in (7.1), but we shall not do that at this time. We may, however, get a good idea of the physical situation by assuming that N-1 of the magnetrons are already locked in phase, and asking what the remaining one will do. Then in (7.6) all the terms from G_2+jB_2 to G_N+jB_N will have the same phase, say ϕ_2. Thus we shall have an equation like (5.4) only now the quantity corresponding to **a** will be equal to (N-1). This very large value will result in an exceedingly strong tendency for the remaining oscillator to lock in with the others. In other words, the more magnetrons there are, the more strongly they will lock in. Of course, the situation when they start, all out of phase with each other, will be complicated. Nevertheless, from the nature of the equations governing them, it is to be assumed that the phases will very soon bring order out of chaos. Each magnetron will try to lock in to the mean phase of all the others, this will tend to build up the strength of this mean, and it can be assumed that in a very short time, a common phase will be established, to which any magnetron which wanders away can be stabilized. The frequency of joint operation will be determined by an extension of Eq. (7.5), a weighted mean of the frequencies of all the magnetrons, weighted by their external Q's. In this synchronized operation, it will of course follow that all the loads, assuming they are matched, will operate in phase with each other, since they are effectively in shunt with each other. Thus these matched loads might well be the inputs to a set of radar antennas, which then will all operate in phase, forming an antenna array.